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The equality of multiple integrals in the series development of a reflection coefficient for waves governed by a general linear differential equation

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Abstract. When wave propagation is governed by a second-order linear differential equation in normal form, the reflection coefficient, when expressed as a power series in terms of a parameter α, consists of a large number of multiple integrals, most of which have been proved to be equal in pairs. The present paper generalises this result to apply to equations of order $2n$, the various stages of the generalisation not being obvious from the case when $n = 1$. The equal pairs of multiple integrals are now only possible for modified integrands, but not for the complete integrands, the number of these pairs of multiple integrals being then independent of the order $2n$.

1. Motivation and introduction

In Darwin's (1924) and Hartree's (1929) investigations into the propagation of electromagnetic waves, each element of the medium was regarded as transmitting fresh waves in all directions, the linear sum of all these wavelets at a point forming the overall wave propagated through the medium. Each newly transmitted wavelet was taken as propagating in free space, regardless of the presence of the medium. White (1942) used these ideas, while Heading (1953, 1963) extended these concepts considerably so as to obtain many formulae relating to propagation in plane-stratified isotropic and anisotropic media. Computer calculations by Westcott (1962a, b, c, d, 1964) based on these formulae gave an insight into the qualitative and quantitative aspects of reflection processes existing in isotropic and anisotropic media.

In these formulae, a reflection coefficient R is expressed as an integral, whose integrand involves the field in the medium together with a function of position that shows by how much the medium differs from free space, this difference being specified by a parameter α. When R is expanded as a power series in α, the coefficients consist of the sum of many multiple integrals, their order being equal to the respective power of α; see Heading (1953, 1963, 1975). When the medium is isotropic, the author later noticed that the two double integrals involved in the coefficient of α^2 are equal, although completely different in explicit form. This led to a detailed investigation by the author (Heading 1981) of the more general case when the field is governed by a general second-order linear differential equation (not necessarily in normal form), the propagation of the newly reradiated waves being governed by an independent second-order linear equation. A parameter α is introduced to describe the difference between the two equations. An examination of the coefficient of α^n in the series development of R (this coefficient consisting of a large number of multiple integrals each of order n) showed that a large number of equal pairs existed amongst these multiple integrals. Formulae were derived giving the number of such pairs, and how they could be recognised both analytically and diagrammatically.

The present paper extends these ideas to linear differential equations of order $2n$, since many physical phenomena are described by equations of order greater than two. The advantages of such a generalisation to order $2n$ have been exploited by the author in other publications; a list is given in Heading (1978, p 281), these investigations throwing up deeper properties than those contained in the simple case $n=1$. Two equations of order $2n$ are specified, the first to govern the propagation of the field under investigation, and the second to govern the propagation of the newly transmitted wavelets (the physical terminology behind the case $n=1$ is imported to describe the general case). Some questions to be answered are: What features of the elementary case are susceptible to generalisation? What is the nature of the generalisation? Are there features of the elementary case that remain for general n, being independent of the order of the differential equation?

The basic ideas behind the simple case $n=1$ are used in this generalisation, but at each stage these basic ideas provide no certain guidance as to the mode of procedure; in fact, for progress to be made, considerable ingenuity is needed throughout the investigation, as can be seen by a careful comparison between the analysis of the present paper and that given in Heading (1981) relating to the elementary case.

2. The differential equations under consideration

The 'carried' wave W is governed by the linear differential operator of even order $2n$ defined by

$$\mathscr{D}_p \equiv D^{2n} + p_{2n-1}(z)D^{2n-1} + \ldots + p_1(z)D + p_0(z), \tag{1}$$

where $D = \mathrm{d}/\mathrm{d}z$. W satisfies the equation $\mathscr{D}_p W = 0$. The functions $p_r(z)$ possess no singularities on the portion of the real z axis under consideration. The adjoint operator is

$$\mathscr{D}_p^{(\mathrm{A})} \equiv D^{2n} - D^{2n-1}p_{2n-1} + \ldots - Dp_1 + p_0;$$

these operators are such that if u and v are any two suitable functions of z, then the Lagrange identity (see Ince 1926)

$$v\mathscr{D}_p u - u\mathscr{D}_p^{(\mathrm{A})} v \equiv D[P_p(u, v)] \tag{2}$$

is satisfied, where $P_p(u, v)$ denotes the bilinear concomitant

$$\begin{aligned} P_p(u, v) \equiv {} & D^{2n-1}u \cdot v + D^{2n-2}u \cdot (-Dv + p_{2n-1}v) \\ & + D^{2n-3}u \cdot [D^2 v - D(p_{2n-1}v) + p_{2n-2}v] + \ldots \\ & + u[-D^{2n-1}v + D^{2n-2}(p_{2n-1}v) - D^{2n-3}(p_{2n-2}v) + \ldots + p_1 v]. \end{aligned} \tag{3}$$

If a discontinuity exists at a point z_0 in any $p_r(z)$ or its derivatives, we impose the boundary conditions that $W, DW, \ldots, D^{2n-1}W$ are all continuous at $z = z_0$.

'Free space' is defined as a particular range of real z in which the coefficients $p_r(z)$ are all constants (the same constants in all such ranges that may exist). If the $2n$

solutions of the polynomial equation

$$\lambda^{2n}+p_{2n-1}\lambda^{2n-1}+\ldots+p_1\lambda+p_0=0$$

are the distinct values α_r, then the 'free space' wave has the form

$$W=\sum_{r=1}^{2n} A_r\,\mathrm{e}^{\alpha_r z}. \tag{4}$$

Consider a second operator $\mathscr{D}_q$ of order $2n$, defined by the coefficients $q_r(z)$, with $\mathscr{D}_q^{(\mathrm{A})}$ as its adjoint. The 'carrier' wave w will be governed by the differential equation using this adjoint operator, namely $\mathscr{D}_q^{(\mathrm{A})}w=0$. We shall define $\mathscr{D}_q$ to be such that in free space $q_r=p_e$ for all coefficients. Hence in free space

$$\mathscr{D}_q^{(\mathrm{A})}w\equiv(D^{2n}-p_{2n-1}D^{2n-1}+\ldots-p_1D+p_0)w=0$$

has the solution

$$w=\sum_{r=1}^{2n} B_r\,\mathrm{e}^{-\alpha_r z}. \tag{5}$$

Define the operator $\mathscr{D}$ of order $2n-1$ to be

$$\mathscr{D}\equiv\mathscr{D}_p-\mathscr{D}_q$$

$$\equiv 0 \text{ in a free space region.}$$

In fact, we shall write

$$p_r(z)-q_r(z)\equiv\alpha y_r(z),$$

where α may be a small parameter; to indicate this, write $\mathscr{D}=\alpha\mathscr{D}_y$.

Consider the Lagrange identity (2) for the two solutions $W(z)$ and $w(z)$:

$$w\mathscr{D}_pW-W\mathscr{D}_p^{(\mathrm{A})}w\equiv D[P_p(W,w)],$$

or

$$w\mathscr{D}_pW-W\alpha\mathscr{D}_y^{(\mathrm{A})}w-W\mathscr{D}_q^{(\mathrm{A})}w\equiv D[P_p(W,w)].$$

Since $\mathscr{D}_pW$ and $\mathscr{D}_q^{(\mathrm{A})}w$ vanish, we obtain upon integration

$$P_p(W,w)|_a^z=-\alpha\int_a^z W\mathscr{D}_y^{(\mathrm{A})}w\,\mathrm{d}t, \tag{6}$$

where $\mathscr{D}_y^{(\mathrm{A})}w$ is of order $2n-1$.

When integrated throughout any free space region, we conclude that $P_p(W,w)=$ constant, since $\mathscr{D}_y=0$.

The left-hand side of (6) contains the $2n$ terms $W, DW, \ldots, D^{2n-1}W$, but the integrand contains W only, together with $w, Dw, \ldots, D^{2n-1}w$. The result is a complete generalisation of the case when $n=1$, given by equation (3) in the paper by Heading (1981). But there is an important difference, for in the integrand only w occurs and not Dw. It is this difference that brings about the distinct results for the generalised theory being developed in this paper. When $n=1$, the two second-order equations were in effect reduced to normal form by means of an integrating factor J, and this specialised treatment ensured (i) that no Dw appeared in the integrand, and (ii) that no reference occurred to adjoint operators, since in normal form the operators are self-adjoint. These specialised features are absent when $n>1$.

3. Evaluation of the bilinear concomitant in free space

We now evaluate $P_p(W, w)$ at a point $z = a$ in free space; matrix notation is an advantage here. Let the column matrix $\boldsymbol{u}$ be defined by

$$\boldsymbol{u} = \begin{pmatrix} u \\ Du \\ D^2u \\ \dots \\ D^{2n-1}u \end{pmatrix}$$

and similarly for any other column matrix. Then from (3),

$$P_p(u, v) = \boldsymbol{u}^T \begin{pmatrix} -D^{2n-1}v + D^{2n-2}(p_{2n-1}v) - \dots + p_1 v \\ \dots \\ -Dv + p_{2n-1}v \\ v \end{pmatrix}$$

$$= \boldsymbol{u}^{\mathrm{T}} \begin{pmatrix} p_1 & -p_2 & p_3 & \dots & p_{2n-1} & -1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ p_{2n-1} & -1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \boldsymbol{v}$$

in free space. Now

$$\boldsymbol{W} = \begin{pmatrix} 1 & 1 & 1 & \dots \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} \begin{pmatrix} A_1\,\mathrm{e}^{\alpha_1 z} \\ A_2\,\mathrm{e}^{\alpha_2 z} \\ A_3\,\mathrm{e}^{\alpha_3 z} \\ \dots \end{pmatrix};$$

hence $P_p(W, w)$ equals

$$(A_1\,\mathrm{e}^{\alpha_1 z} A_2\,\mathrm{e}^{\alpha_2 z} \dots) \begin{pmatrix} 1 & \alpha_1 & \dots \\ 1 & \alpha_2 & \dots \\ \vdots & \vdots & \dots \end{pmatrix} \begin{pmatrix} p_1 & p_2 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ p_{2n-1} & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots \\ \alpha_1 & \alpha_2 & \dots \\ \vdots & \vdots & \dots \end{pmatrix} \begin{pmatrix} B_1\,\mathrm{e}^{-\alpha_1 z} \\ B_2\,\mathrm{e}^{-\alpha_2 z} \\ \dots \end{pmatrix},$$

where the various minus signs have been taken into account. Generally, consider the product

$$(1 \quad \alpha \quad \alpha^2 \quad \dots) \begin{pmatrix} p_1 & p_2 & \dots & 1 \\ p_2 & p_3 & \dots & 0 \\ p_3 & p_4 & \dots & 0 \\ \vdots & \vdots & \dots & \end{pmatrix} \begin{pmatrix} 1 \\ \beta \\ \beta^2 \\ \dots \end{pmatrix}$$

for any two roots α and β. The product equals

$$\begin{aligned} 1(p_1 + p_2\beta + p_3\beta^2 + \dots) &+ \alpha(p_2 + p_3\beta + p_4\beta^2 + \dots) + \alpha^2(p_3 + p_4\beta + p_5\beta^2 + \dots) + \dots \\ &= p_1 + p_2(\alpha + \beta) + p_3(\alpha^2 + \alpha\beta + \beta^2) + \dots \\ &= (\alpha - \beta)^{-1}[p_1(\alpha - \beta) + p_2(\alpha^2 - \beta^2) + p_3(\alpha^3 - \beta^3) + \dots] \\ &= (\alpha - \beta)^{-1}(-p_0 + p_0) = 0 \qquad \text{provided } \alpha \neq \beta, \end{aligned}$$

but when $\alpha=\beta$, the product equals

$$p_1+2p_2\alpha+3p_3\alpha^2+\ldots=\mathrm{d}S/\mathrm{d}\alpha|_\alpha\equiv S'(\alpha)$$

say, where

$$S\equiv\alpha^n+p_{2n-1}\alpha^{2n-1}+\ldots+p_1\alpha+p_0.$$

Hence in free space

$$P_p(W,w)=A_1B_1S'(\alpha_1)+A_2B_2S'(\alpha_2)+\ldots=\sum_{r=1}^{2n}A_rB_rS'(\alpha_r).$$

In another free space region containing the point $z=b>a$ (with the same roots α_r), let

$$W=\sum_{r=1}^{2n}\bar{A}_r\,\mathrm{e}^{\alpha_r z},\qquad w=\sum_{r=1}^{2n}\bar{B}_r\,\mathrm{e}^{-\alpha_r z}.$$

Then result (6), with z replaced by b, becomes

$$\sum_{r=1}^{2n}(\bar{A}_r\bar{B}_r-A_rB_r)S'(\alpha_r)=-\alpha\int_a^b W\mathscr{D}_y^{(\mathrm{A})}w\,\mathrm{d}t. \tag{7}$$

This is a relation between the coefficients in one free space region and those in another, expressed in terms of an integral throughout the intervening space of the carried and carrier waves (and the derivatives of w up to order $2n-1$). In particular, if the parameter α vanishes, in which case W and w satisfy adjoint equations respectively,

$$\sum_{r=1}^{2n}(\bar{A}_r\bar{B}_r-A_rB_r)S'(\alpha_r)=0,$$

a relation connecting the coefficients of the two fields in two distinct free space regions.

4. The integral equation for *W*

Equation (6) forms an integro-differential equation for W, where W alone occurs on the right-hand side, but $W, DW,\ldots,D^{2n-1}W$ on the left-hand side. Let $w_1, w_2,\ldots,w_{2n}$ denote $2n$ independent solutions of $\mathscr{D}_q^{(\mathrm{A})}w=0$. In the free space region containing $z=a$, let

$$w_s=\sum_{r=1}^{2n}B_r^{(s)}\,\mathrm{e}^{-\alpha_r z},$$

while in the free space region containing $z=b$, let

$$w_s=\sum_{r=1}^{2n}\bar{B}_r^{(s)}\,\mathrm{e}^{-\alpha_r z},$$

with $2(2n)^2$ coefficients totally. Then (6) yields $2n$ equations

$$P_p(W,w_s)|_a^z=-\alpha\int_a^z W\mathscr{D}_y^{(\mathrm{A})}w_s\,\mathrm{d}t,\qquad s=1,2,\ldots,2n,$$

or, from (3),

$$W[-D^{2n-1}w_s+D^{2n-2}(p_{2n-1}w_s)-\ldots+p_1w_s]+\ldots+D^{2n-2}W\cdot(-Dw_s+p_{2n-1}w_s)$$
$$+D^{2n-1}W\cdot w_s-\sum_{r=1}^{2n}A_rB_r^{(s)}S'(\alpha_r)+\alpha\int_a^z W\mathscr{D}_y^{(A)}w_s\,dt=0.$$

These equations enable us to obtain an integral equation involving W only without its derivatives, namely,

$$W=-M/N,$$

where

$$M=(-1)^{n-1}\begin{vmatrix} D^{2n-2}w_1-D^{2n-3}(p_{2n-1}w_1)+\ldots+p_2w_1 & \ldots & -Dw_1+p_{2n-1}w_1 \\ & & w_1 \quad -\sum_{r=1}^{2n}A_rB_r^{(1)}S'(\alpha_r)+\alpha\int_a^z W\mathscr{D}_y^{(A)}w_1\,dt \\ D^{2n-2}w_2-D^{2n-3}(p_{2n-1}w_2)+\ldots+p_2w_2 & \ldots & -Dw_2+p_{2n-1}w_2 \\ & & w_2 \quad -\sum_{r=1}^{2n}A_rB_r^{(2)}S'(\alpha_r)+\alpha\int_a^z W\mathscr{D}_y^{(A)}w_2\,dt \\ \ldots & & \ldots \end{vmatrix}$$

$$=\begin{vmatrix} D^{2n-2}w_1 & D^{2n-3}w_1 & \ldots & Dw_1 & w_1 & -\sum_{r=1}^{2n}A_rB_r^{(1)}S'(\alpha_r)+\alpha\int_a^z W\mathscr{D}_y^{(A)}w_1\,dt \\ D^{2n-2}w_2 & D^{2n-3}w_2 & \ldots & Dw_2 & w_2 & -\sum_{r=1}^{2n}A_rB_r^{(2)}S'(\alpha_r)+\alpha\int_a^z W\mathscr{D}_y^{(A)}w_2\,dt \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \end{vmatrix}$$

and N is the Wronksian

$$N=\begin{vmatrix} D^{2n-1}w_1 & D^{2n-2}w_1 & \ldots & Dw_1 & w_1 \\ D^{2n-1}w_2 & D^{2n-2}w_2 & \ldots & Dw_2 & w_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \end{vmatrix}$$

upon simplification in each case.

We have the identity

$$w_s\mathscr{D}_qu-u\mathscr{D}_q^{(A)}w_s\equiv D[P_q(u,w_s)].$$

Let u denote a solution of $\mathscr{D}_qu=0$, while we have already chosen w_s to denote $2n$ independent solutions of $\mathscr{D}_q^{(A)}w=0$. Hence we have the $2n$ equations

$$P_q(u,w_s)=C_s$$

for $u, Du, \ldots, D^{2n-1}u$. The choice of $2n$ independent columns C_s will give $2n$ solutions u_s. In particular, if successively all the C_s are zero except one chosen to equal unity, then we define a special set of solutions $u_1, u_2, \ldots, u_{2n}$, each being the ratio of two determinants, the denominator being N and the numerators the *cofactors* respectively of the last column of M.

Thus we may write the integral equation of the second kind as

$$W=\sum_{s=1}^{2n} u_s\left(\sum_{r=1}^{2n} A_r B_r^{(s)} S'(\alpha_r)-\alpha \int_a^z W \mathscr{D}_y^{(\mathrm{A})} w_s \,\mathrm{d}t\right). \tag{8}$$

In other words, W is now expressed in terms of the u_s (the $2n$ special solutions of the adjoint of the equation that we have called the carrier equation), which in turn are expressed in terms of the w_s, these latter solutions appearing in the integrands.

For any particular value of s, from (7) we may write

$$\sum_{r=1}^{2n} A_r B_r^{(s)} S'(\alpha_r)-\alpha \int_a^z W \mathscr{D}_y^{(\mathrm{A})} w_s \,\mathrm{d}t=\sum_{r=1}^{2n} \bar{A}_r \bar{B}_r^{(s)} S'(\alpha_r)+\alpha \int_z^b W \mathscr{D}_y^{(\mathrm{A})} w_s \,\mathrm{d}t, \tag{9}$$

implying that, of the $2n$ integrals occurring in the integral equation, some may be taken from a to z, and others from z to b. At the moment, the selection of the limits is arbitrary, though certain demands to be made later will fix the choice.

5. The choice of the carrier waves w_s

Successive substitution in (8) will yield a formal development of W in terms of α, and the subsequent substitution of this into (7) will finally yield a development of the left-hand side in terms of α without the necessity of solving equation (1) for W. The coefficient of α^m in W involves many multiple integrals of order m, and so does the development of (7) for $\Sigma(\bar{A}_r\bar{B}_r - A_rB_r)S'(\alpha_r)$ for a particular choice $w = w_s$. The intention is to simplify this complicated summation involving the coefficients, so that only one coefficient appearing in W remains to be calculated. Solutions $w_1, w_2, \ldots, w_{2n}$ are also specially selected so that most terms A_rB_r, $\bar{A}_r\bar{B}_r$ in (8) or (9) vanish; integrals either from a to z or from z to b are taken so as to achieve this.

In particular, if $n = 1$, and if the second-order equation is in normal form, the integral equation simplifies, since $u_1 = -w_2/N$, $u_2 = w_1/N$, where $N = \text{constant}$. The theory then reverts to the case already studied by Heading (1981).

The selection of the coefficients in W and w_s at $z = a$ and $z = b$ can be more easily grasped by considering the simple cases when $n = 1, 2, 3$ as in the tables below. The procedure underlying these simple cases can then be applied to the general case. For any wave, it is arbitrary which n exponential solutions in free space are regarded as 'upgoing' and which n as 'downgoing' except that the descriptions for W are reversed for all the w_s (since the signs in the exponential indices are all reversed). For any w_s for general n, there are $4n$ coefficients in total. Of these, $2n-1$ can be chosen to vanish, and one can equal unity; the particular solution is then defined uniquely. When necessary for our purpose, this special unit coefficient will be attached to an 'incident' wave, either 'upgoing' at $z = a$, or 'downgoing' at $z = b$. The remaining coefficients will be either reflection or transmission coefficients, and the various symbols R, T, r, t (incidence from below) and r', t' (incidence from above) are used when necessary; otherwise the A's and B's are used as previously. Throughout, we choose w_1 to appear in (7).

$n = 1$:

W	$^{\uparrow}1$	$^{\downarrow}R$	$^{\uparrow}T$	$^{\downarrow}0$	Integral in (8) or (9)
w_1	$^{\downarrow}r$	$^{\uparrow}1$	$^{\downarrow}0$	$^{\uparrow}t$	z to b
w_2	t'	0	1	r'	a to z

$n = 2$:

W	$^{\uparrow}1$	$^{\downarrow}R_1$	$^{\uparrow}0$	$^{\downarrow}R_2$	$^{\uparrow}T_1$	$^{\downarrow}0$	$^{\uparrow}T_2$	$^{\downarrow}0$	Integral
w_1	$^{\downarrow}r$	$^{\uparrow}1$	$^{\downarrow}B_3^{(1)}$	$^{\uparrow}0$	$^{\downarrow}0$	$^{\uparrow}\bar{B}_2^{(1)}$	$^{\downarrow}0$	$^{\uparrow}\bar{B}_4^{(1)}$	z to b
w_2	$B_1^{(2)}$	$B_2^{(2)}$	$B_3^{(2)}$	$B_4^{(2)}$	0	$\bar{B}_2^{(2)}$	0	0	z to b
w_3	t'	0	0	0	$\bar{B}_1^{(3)}$	$\bar{B}_2^{(3)}$	$\bar{B}_3^{(3)}$	$\bar{B}_4^{(3)}$	a to z
w_4	0	0	$B_3^{(4)}$	0	$\bar{B}_1^{(4)}$	$\bar{B}_2^{(4)}$	$\bar{B}_3^{(4)}$	$\bar{B}_4^{(4)}$	a to z

$n = 3$:

W	$^{\uparrow}1$	$^{\downarrow}R_1$	$^{\uparrow}0$	$^{\downarrow}R_2$	$^{\uparrow}0$	$^{\downarrow}R_3$	$^{\uparrow}T_1$	$^{\downarrow}0$	$^{\uparrow}T_2$	$^{\downarrow}0$	$^{\uparrow}T_3$	$^{\downarrow}0$	Integral
w_1	$^{\downarrow}r$	$^{\downarrow}1$	$^{\downarrow}B_3^{(1)}$	$^{\uparrow}0$	$^{\downarrow}B_5^{(1)}$	$^{\uparrow}0$	$^{\downarrow}0$	$^{\uparrow}\bar{B}_2^{(1)}$	$^{\downarrow}0$	$^{\uparrow}\bar{B}_4^{(1)}$	$^{\downarrow}0$	$^{\uparrow}\bar{B}_6^{(1)}$	z to b
w_2	$B_1^{(2)}$	$B_2^{(2)}$	$B_3^{(2)}$	$B_4^{(2)}$	$B_5^{(2)}$	$B_6^{(2)}$	0	$\bar{B}_2^{(2)}$	0	0	0	0	z to b
w_3	$B_1^{(3)}$	$B_2^{(3)}$	$B_3^{(3)}$	$B_4^{(3)}$	$B_5^{(3)}$	$B_6^{(3)}$	0	0	0	$\bar{B}_4^{(3)}$	0	0	z to b
w_4	t'	0	0	0	0	0	$\bar{B}_1^{(4)}$	$\bar{B}_2^{(4)}$	$\bar{B}_3^{(4)}$	$\bar{B}_4^{(4)}$	$\bar{B}_5^{(4)}$	$\bar{B}_6^{(4)}$	a to z
w_5	0	0	$B_3^{(5)}$	0	0	0	$\bar{B}_1^{(5)}$	$\bar{B}_2^{(5)}$	$\bar{B}_3^{(5)}$	$\bar{B}_4^{(5)}$	$\bar{B}_5^{(5)}$	$\bar{B}_6^{(5)}$	a to z
w_6	0	0	0	0	$B_5^{(6)}$	0	$\bar{B}_1^{(6)}$	$\bar{B}_2^{(6)}$	$\bar{B}_3^{(6)}$	$\bar{B}_4^{(6)}$	$\bar{B}_5^{(6)}$	$\bar{B}_6^{(6)}$	a to z

For W (below), the $2n$ coefficients A_r are chosen as follows. One incident wave has unit amplitude, the other $n-1$ incident coefficients being zero. The n reflection coefficients $R_1, \ldots, R_n$ are attached to the n downgoing waves. For W (above), the n downgoing waves are given zero coefficients, while there are n transmission coefficients $T_1, \ldots, T_n$. Overall, $2n-1$ coefficients must be zero, while one must be unity. The whole arrangement defines a unique solution.

The $2n$ solutions w_s are chosen so as to be independent. Each has $2n-1$ zero coefficients, with $2n+1$ incident, reflected and transmitted waves. The choice is made so that for each w_s (except $s = n+1$) one of

$$\sum_{r=1}^{2n} A_r B_r^{(s)} S'(\alpha_r) \qquad \text{and} \qquad \sum_{r=1}^{2n} \bar{A}_r \bar{B}_r^{(s)} S'(\alpha_r)$$

is zero. If the former, then the integral $\int_a^z$ is used in (8), while if the latter, then $\int_z^b$ is used as in (9).

w_1 (below) has a unit incident wave, the first reflection coefficient r, $n-1$ further reflection coefficients, with zero coefficients for the remaining $n-1$ incident waves. Above, there are n zero downgoing waves, and n transmission coefficients. Hence $\int_z^b$ is used. $w_2, \ldots, w_n$ (below) each have $2n$ non-zero coefficients. Above, each has only one non-zero coefficient (corresponding respectively to the zeros in W, the last one being discounted). Hence $\int_z^b$ is used. w_{n+1} (below) had t' only in the first column; above, all the $2n$ coefficients occur, and $\int_a^z$ is used, though $\Sigma A_r B_r^{(n+1)} S'(\alpha_r)$ (below) is not zero, the only case thus chosen. Finally, $w_{n+2}, \ldots, w_{2n}$ each have only one non-zero coefficient below, corresponding to the $n-1$ zeros in W. Above, all the $2n$ coefficients are non-zero, and $\int_a^z$ is used.

Slight alterations in this distribution may be allowed, but apart from these, the general pattern is unique so as to achieve our objective. Certainly the simple case given by $n=1$ is thereby generalised.

With this choice, integral equation (8) becomes

$$W = t'S'(\alpha_1)u_{n+1} + \alpha \sum_{s=1}^{n} u_s \int_z^b W\mathscr{D}_y^{(A)} w_s \, dt - \alpha \sum_{s=n+1}^{2n} u_s \int_a^z W\mathscr{D}_y^{(A)} w_s \, dt, \tag{10}$$

while result (7)

$$\sum_{r=1}^{2n} (\bar{A}_r\bar{B}_r^{(1)} - A_rB_r^{(1)})S'(\alpha_r) = -\alpha \int_a^b W\mathscr{D}_y^{(A)} w_1 \, dt$$

reduces to

$$R_1 = -\frac{S'(\alpha_1)}{S'(\alpha_2)} r + \frac{\alpha}{S'(\alpha_2)} \int_a^b W\mathscr{D}_y^{(A)} w_1 \, dt. \tag{11}$$

These are complete generalisations of the simpler formulae given when $n=1$ by Heading (1981).

The development of (10) as a series in α commences with

$$W \doteqdot t'S'(\alpha_1)u_{n+1},$$

giving

$$R_1 \doteqdot -\frac{S'(\alpha_1)}{S'(\alpha_2)} r + \frac{\alpha t' S'(\alpha_1)}{S'(\alpha_2)} \int_a^b u_{n+1}\mathscr{D}_y^{(A)} w_1 \, dt, \tag{12}$$

expressing R_1 to $O(\alpha)$, valid when α is small, namely when operator (1) possesses coefficients differing only slightly from those in $\mathscr{D}_q$.

6. The number of equal multiple integrals

The consideration of the various multiple integrals produced by successive substitution now becomes rather involved. As we illustrate the procedure for dealing with the equations and results for general values of n, it should be pointed out that the simpler case $n=2$ contains all the necessary ingredients to distinguish the general case from the specially simple case when $n=1$ in normal form.

As in Heading (1981), the investigation is carried out by means of the reversal of the order of integration in the multiple integral of general order:

$$\int_a^b f(z)\, dz \int_a^z g(y)\, dy \int_y^b \ldots \int_x^b h(w)\, dw \int_a^w j(v)\, dv$$
$$= \int_a^b j(z)\, dz \int_z^b h(y)\, dy \int_a^y \ldots \int_a^x g(w)\, dw \int_w^b f(v)\, dv,$$

where

(i) the functions in the integrands are reversed in order;

(ii) the integrals are reversed in order (not counting the first integral), with the lower limit a being replaced by the upper limit b, and vice versa;

(iii) any permutation of these limits is permissible. To avoid the unnecessary writing down of the variables of integration, no confusion can arise if we write this identity as

$$\int_a^b f \int_a g \int^b \ldots \int^b h \int_a j = \int_a^b j \int^b h \int_a \ldots \int_a g \int^b f. \tag{13}$$

This applies for any permutation of limits a (lower) and b (upper).

Define the operator

$$\mathscr{P}_s = \begin{cases} u_s \int^b \mathscr{D}_y^{(A)} w_s, & 1 \leqslant s < n \\ -u_s \int_a \mathscr{D}_y^{(A)} w_s, & n+1 < s \leqslant 2n, \end{cases}$$

the variable of integration and the variable limit being given by the context of any equation. Then the equation (10) for W is

$$W = t'S'(\alpha_1)u_{n+1} + \alpha \sum_{s=1}^{2n} \mathscr{P}_s W.$$

The development of W by successive substitution yields

$$t'S'(\alpha_1) \sum \mathscr{P}_s u_{n+1}$$

as the coefficient of α, and

$$t'S'(\alpha_1)\Big(\sum \mathscr{P}_s\Big)^m u_{n+1}$$

as the coefficient of α^m. Hence the coefficient of α^{m+1} in R_1 is

$$\frac{t'S'(\alpha_1)}{S'(\alpha_2)} \int_a^b \mathscr{D}_y^{(A)} w_1 \Big(\sum \mathscr{P}_s\Big)^m u_{n+1}.$$

The question of the convergence of the series for R_1 is an issue that is independent of the investigation of relationships between integrals produced by the formal development.

There will be n^m permutations involved in $(\Sigma\, \mathscr{P}_s)^m$, yielding n^m multiple integrals. In the consideration of any particular multiple integral, often only a modified integral containing a selection of terms from the integrand may have to be examined, since $\mathscr{D}_y^{(A)} w_1$, u_{n+1}, and so on, all contain many terms. Our investigation must show how many of these integrals (with modified integrands if necessary) are equal in pairs—yielding dual or reciprocal pairs—and how many in themselves form self-dual integrals.

When $n = 1$, the author's previous investigation (Heading 1981) was more direct, since $\mathscr{D}_y^{(A)} w_s$ consisted of one term only, involving w_s, while u_s also consisted of one term only (involving either w_1 or w_2).

In the general case, for brevity denote $\mathscr{D}_y^{(A)} w_s$ by $\bar{w}_s$, a linear expression in w_s and its derivatives up to order $2n-1$. A modified integrand will only require the use of the one term not involving a derivative: we shall denote this by

$$\text{modified } \bar{w}_s \equiv \bar{w}_s^\circ \equiv Y w_s.$$

We have defined u_s as

$$u_s=\frac{(-1)^s}{N}\begin{vmatrix} D^{2n-2}w_1 & \dots & Dw_1 & w_1 \\ \vdots & \dots & \vdots & \vdots \\ D^{2n-2}w_{2n} & \dots & Dw_{2n} & w_{2n} \end{vmatrix} \quad \text{(with the row containing } w_s \text{ omitted).}$$

When expanded, this consists of $2n-1$ terms as $w_r(s,r)$ $s\neq r$, the expansion taking place down the last column, where $(s,r)\equiv(r,s)$ denotes the determinant of order $2n-2$,

$$\begin{vmatrix} D^{2n-2}w_1 & \dots & Dw_1 \\ \vdots & \dots & \vdots \\ D^{2n-2}w_{2n} & \dots & Dw_{2n} \end{vmatrix}$$

(with rows s and r omitted) divided by N.

It will transpire that only $s=1$ and $n+1$ will be needed, with $r=n+1$ and 1 respectively, in modified integrals. Then

$$\text{modified } u_1\equiv u_1^\circ=(-1)^n w_{n+1}(1,n+1),$$
$$\text{modified } u_{n+1}\equiv u_{n+1}^\circ=(-1)^{n+1}w_1(n+1,1),$$

with identical coefficients $(1, n+1)$ ($\equiv Z$, say) appearing in both modifications.

We now consider any multiple integral of order m

$$\int_a^b \bar{w}_1(\mathscr{P}_s\mathscr{P}_t\dots)u_{n+1}.$$

The order of integration is reversed in keeping with result (13), and the result is compared with another permutation to see if equality is possible, the $\bar{w}$'s and the u's being modified if necessary in the process. A careful and detailed examination of this process reveals that $\bar{w}_1$ at the beginning and u_{n+1} at the end place a restriction on what is possible: permutations containing only $\mathscr{P}_1$ and $\mathscr{P}_{n+1}$ are allowed. This means that all permutations are allowed in the simplest case when $n=1$.

An example will now show what is involved. Take the multiple integral of order eight that would arise in the coefficient of α^8 in R_1:

$$\int_a^b \bar{w}_1\mathscr{P}_1\mathscr{P}_1\mathscr{P}_N\mathscr{P}_1\mathscr{P}_N\mathscr{P}_N\mathscr{P}_1u_N \tag{14}$$

where the suffix N denotes $n+1$ for brevity. Explicitly, this is

$$\int_a^b \bar{w}_1u_1\int^b \bar{w}_1u_1\int^b \bar{w}_1u_N\int_a \bar{w}_Nu_1\int^b \bar{w}_1u_N\int_a \bar{w}_Nu_N\int_a \bar{w}_Nu_1\int^b \bar{w}_1u_N. \tag{15}$$

This is equal to the reversed integral

$$\int_a^b \bar{w}_1u_N\int_a \bar{w}_Nu_1\int^b \bar{w}_Nu_N\int^b \bar{w}_1u_N\int_a \bar{w}_Nu_1\int^b \bar{w}_1u_N\int_a \bar{w}_1u_1\int_a \bar{w}_1u_1. \tag{16}$$

This is not a permissible permutation; in fact, it is not a permutation of the defined operators at all. A direct equality is not possible in this case. But if certain of the $\bar{w}$'s and u's are modified in keeping with the above definitions, we have

$$\int_a^b \bar{w}_1 u_N \int_a \bar{w}_N u_1 \int^b \bar{w}_N^\circ u_N^\circ \int^b \bar{w}_1 u_N \int_a \bar{w}_N u_1 \int^b \bar{w}_1 u_N \int_a \bar{w}_1^\circ u_1^\circ \int_a \bar{w}_1^\circ u_1^\circ$$

$$= \int_a^b \bar{w}_1 u_N \int_a \bar{w}_N u_1 \int^b Y w_N (-1)^N w_1 Z \int^b \bar{w}_1 u_N \int_a \bar{w}_N u_1 \int^b \bar{w}_1 u_N$$

$$\times \int_a Y w_1 (-1)^n w_N Z \int_a Y w_1 (-1)^n w_N Z. \qquad (17)$$

Where necessary this is altered slightly so that the individual symbols are appropriately in juxtaposition to the integral signs:

$$\int_a^b \bar{w}_1 u_N \int_a \bar{w}_N u_1 \int^b Y w_1 (-1)^N w_1 Z \int^b \bar{w}_1 u_N$$

$$\times \int_a \bar{w}_N u_1 \int^b \bar{w}_1 u_N \int_a Y w_N (-1)^n w_1 Z \int_a Y w_N (-1)^n w_1 Z.$$

It can now be seen that this is the modified form of another permutation (the limits of integration dictating what symbols must be written down), namely

$$-\int_a^b \bar{w}_1 u_N \int_a \bar{w}_N u_1 \int^b \bar{w}_1^\circ u_1^\circ \int^b \bar{w}_1 u_N \int_a \bar{w}_N u_1 \int^b \bar{w}_1 u_N \int_a \bar{w}_N^\circ u_N^\circ \int_a \bar{w}_N^\circ u_N^\circ \qquad (18)$$

originating from the permutation

$$-\int_a^b \bar{w}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_N \mathscr{P}_N u_N. \qquad (19)$$

It can now be seen what modifications are necessary in (14) and (19) when expressed as (17) and (18). If two identical suffixes occur between two integral signs (for example, $\bar{w}_1 u_1$, $\bar{w}_1 u_1$, $\bar{w}_N u_N$ in (15), and the same in (18)), all such symbols must be modified. To enable us to denote this easily, let

$$\mathscr{P}_s^\circ \equiv (\pm) u_s \int \bar{w}_s^\circ, \qquad {}^\circ\mathscr{P}_s \equiv (\pm) u_s^\circ \int \bar{w}_s, \qquad {}^\circ\mathscr{P}_s^\circ \equiv (\pm) u_s^\circ \int \bar{w}_s^\circ;$$

then the equality just deduced may be expressed in the form

$$\int_a^b \bar{w}_1^\circ\, {}^\circ\mathscr{P}_1^\circ\, {}^\circ\mathscr{P}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_N^\circ\, {}^\circ\mathscr{P}_N \mathscr{P}_1 u_N = -\int_a^b \bar{w}_1 \mathscr{P}_N \mathscr{P}_1^\circ\, {}^\circ\mathscr{P}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_N^\circ\, {}^\circ\mathscr{P}_N^\circ u_N^\circ,$$

the sign being $(-1)^L$, where $2L$ = total number of modified $\bar{w}$'s and u's in either integral.

Although for ease of explanation this result has been proved for a particular permutation, it is valid for any permutation.

Some permutations may be self-dual, and need no modification. For example,

$$\int_a^b \bar{w}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_N \mathscr{P}_1 u_N$$

yields itself upon reversal. Other self-dual permutations require modification, as

$$\int_a^b \bar{w}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_1 \mathscr{P}_N \mathscr{P}_N \mathscr{P}_1 u_N;$$

the modified integral, equal to itself upon reversal, is

$$\int_a^b \bar{w}_1 \mathscr{P}_N \mathscr{P}_1^\circ {}^\circ\mathscr{P}_1 \mathscr{P}_N^\circ {}^\circ\mathscr{P}_N \mathscr{P}_1 u_N.$$

There are no self-dual integrals (modified when necessary) of even order.

Some permutations may need only two modifications, such as

$$\int_a^b \bar{w}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_N^\circ u_N^\circ = -\int_a^b \bar{w}_1^\circ {}^\circ\mathscr{P}_1 \mathscr{P}_N \mathscr{P}_1 \mathscr{P}_N \mathscr{P}_1 u_N.$$

There cannot be a permutation that requires modification of every symbol, though permutations will exist that require modification of all but two symbols:

$$\int_a^b \bar{w}_1^\circ {}^\circ\mathscr{P}_1^\circ {}^\circ\mathscr{P}_1^\circ {}^\circ\mathscr{P}_1 u_N = -\int_a^b \bar{w}_1 \mathscr{P}_N^\circ {}^\circ\mathscr{P}_N^\circ {}^\circ\mathscr{P}_N^\circ u_N^\circ.$$

Since only two suffixes, 1 and N, enter these pairs, the number of dual pairs and self-dual integrals (modified when necessary) for each value of n will be the same as in the case $n=1$ (in which case no modification is needed when an integrating factor effectively reduces the two equations to normal form). These numbers have been tabulated in Heading (1981). Consequently the generalisation consists, not in the number of such integrals, but in the analytical forms of W and R_1 (and the special choice of the w's and u's), and in the modifications to the integrands necessary to achieve identity (perhaps with a minus sign). In the diagram, the results when $n=2$ are illustrated in the same way as adopted for the case when $n=1$, except that a lot of redundant integrals are produced that do not participate in the theorem (represented by short lines that do not lead to further branches).

The diagram shows that the equality relating to the coefficient α^m contains the sign $(-1)^m$. This may be proved generally.

Writing down only the suffixes contained in such an integral, we may have

$$1 \quad a \quad b \quad c \quad \ldots \quad f \quad 3,$$

where there are m suffixes $a, b, c, \ldots, f$, denoting either 1 or 3. If $f=1$, let N equal the number of times that two equal suffixes stand side by side. When m is increased to $m+1$, two possibilities arise:

$$1 \quad a \quad b \quad c \quad \ldots \quad 1 \quad {*1} \quad 3$$

$$1 \quad a \quad b \quad c \quad \ldots \quad 1 \quad 3 \quad {*3}.$$

In both cases, an extra pair is created, denoted by *, so that the number of pairs is now $N+1$. But if $f=3$, let M denote the number of such pairs in

$$1 \quad a \quad b \quad c \quad \ldots \quad 3 \quad 3.$$

Increasing m to $m+1$, we have two possibilities

$$1 \quad a \quad b \quad c \quad \ldots \quad 3 \quad {*1} \quad {*3}$$

$$1 \quad a \quad b \quad c \quad \ldots \quad 3 \quad {*3} \quad {*3}.$$

In the first one, a pair is broken at **, while in the second one two pairs arise from one pair at **. The number of such pairs is therefore $M-1$ or $M+1$ respectively.

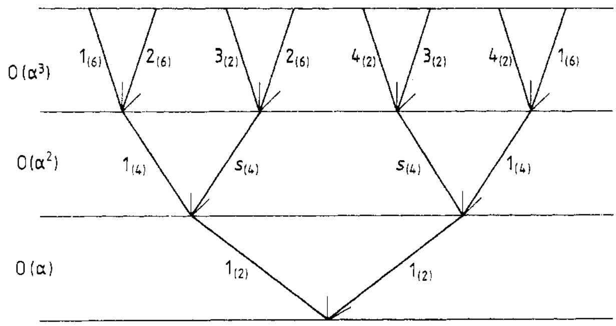

Figure 1. A branch to the left represents the operator $\mathcal{P}_1$, while one to the right represents $\mathcal{P}_3$. Equal numbers in a horizontal band denote equality (with a ± sign as appropriate). The subscripts denote the number of modifications necessary in the integrands. The band $O(\alpha^m)$ refers to W, and $O(\alpha^{m+1})$ to R_1. The equal integrals are:

$O(\alpha)$ $1_{(2)}$: $\int \bar{w}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 u_3 = -\int \bar{w}_1 \mathcal{P}_3^{\circ} u_3^{\circ}$

$O(\alpha^3)$ $1_{(6)}$: $\int \bar{w}_1^{\circ}\,{}^{\circ}\mathcal{P}_1^{\circ}\,{}^{\circ}\mathcal{P}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 u_3 = -\int \bar{w}_1 \mathcal{P}_3^{\circ}\,{}^{\circ}\mathcal{P}_3^{\circ}\,{}^{\circ}\mathcal{P}_3^{\circ} u_3^{\circ}$

$O(\alpha^2)$ $1_{(4)}$: $\int \bar{w}_1^{\circ}\,{}^{\circ}\mathcal{P}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 u_3 = \int \bar{w}_1 \mathcal{P}_3^{\circ}\,{}^{\circ}\mathcal{P}_3^{\circ} u_3^{\circ}$

$2_{(6)}$: $\int \bar{w}_1^{\circ}\,{}^{\circ}\mathcal{P}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 \mathcal{P}_3^{\circ} u_3^{\circ} = -\int \bar{w}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 \mathcal{P}_3^{\circ}\,{}^{\circ}\mathcal{P}_3^{\circ} u_3^{\circ}$

$s_{(4)}$: $\int \bar{w}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 \mathcal{P}_3^{\circ} u_3^{\circ}$ (self-dual)

$3_{(2)}$: $\int \bar{w}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 \mathcal{P}_3 \mathcal{P}_1 u_3 = -\int \bar{w}_1 \mathcal{P}_3 \mathcal{P}_1 \mathcal{P}_3^{\circ} u_3^{\circ}$

$s_{(0)}$: $\int \bar{w}_1 \mathcal{P}_3 \mathcal{P}_1 u_3$ (self-dual)

$4_{(2)}$: $\int \bar{w}_1 \mathcal{P}_3 \mathcal{P}_1^{\circ}\,{}^{\circ}\mathcal{P}_1 u_3 = -\int \bar{w}_1 \mathcal{P}_3^{\circ}\,{}^{\circ}\mathcal{P}_3 \mathcal{P}_1 u_3.$

When $m = 1$, there are only two cases: 1 1 3 and 1 3 3, where $N = 1$ and $M = 1$. It follows that for general m the number of such pairs is even or odd when m is even or odd respectively. The signs relating to the coefficient α^m are therefore always $(-1)^m$.

Finally, if we define $\mathcal{R}_3 = -\mathcal{P}_3$, we note that the number of $\mathcal{P}_3$'s occurring in an equality in the coefficient of α^m equals m. Replacing $\mathcal{P}_3$ by $\mathcal{R}_3$, we see immediately that all signs in the equalities become positive throughout.

References

Darwin C G 1924 *Phil. Trans. Camb. Phil. Soc.* **23** 137–67
Hartree D R 1929 *Proc. Camb. Phil. Soc.* **25** 97–120
Heading J 1953 *PhD Thesis* University of Cambridge
—— 1963 *J. Res. NBS* **67D** 65–77
—— 1975 *Ordinary Differential Equations, Theory and Practice* (London: Elek Science)
—— 1978 *Proc. R. Soc. Lond.* A **360** 279–300
—— 1981 *J. Phys. A: Math. Gen.* **14** 357–67

Ince E L 1926 *Ordinary Differential Equations* (London: Longmans, Green)
Westcott B S 1962a *J. Atmos. Terr. Phys.* **24** 385–99
—— 1962b *J. Atmos. Terr. Phys.* **24** 619–31
—— 1962c *J. Atmos. Terr. Phys.* **24** 701–13
—— 1962d *J. Atmos. Terr. Phys.* **24** 921–36
—— 1964 *J. Atmos. Terr. Phys.* **26** 341–50
White F W G 1942 *Electromagnetic Waves* (London: Methuen)